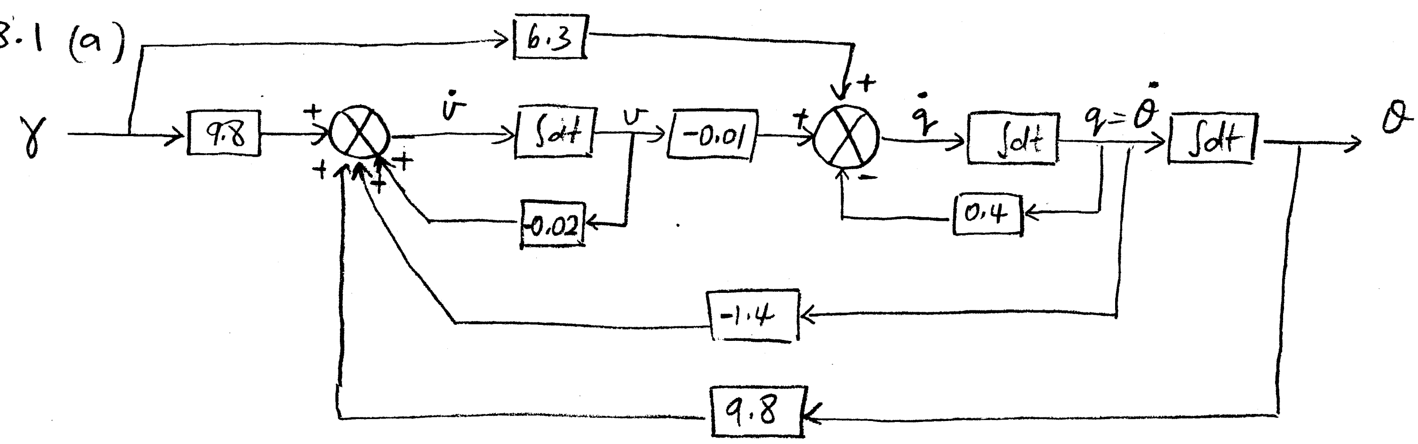
S8.1 (a)



(b) $$\det[sI-A] = \det\begin{bmatrix} s+0.4 & 0 & 0.01 \\ -1 & s & 0 \\ 1.4 & -9.8 & s+0.02 \end{bmatrix}$$

$$= (s+0.4)[s(s+0.02)] + 0.01[9.8-1.4s]$$

$$= s^3 + 0.42s^2 - 0.006s + 0.098$$

C.E. : $s^3 + 0.42s^2 - 0.006s + 0.098 = 0$

(c) Eigenvalues; $\lambda_1 = -0.66$, $\lambda_{2,3} = 0.12 \pm 0.37j$

(d) The system is unstable, since the complex eigenvalues have positive real part.

(e) $(s+2)(s+1-j)(s+1+j) = (s+2)(s^2+2s+2) = s^3+4s^2+6s+4$

C.E. : $s^3+4s^2+6s+4 = 0$

(a) 1. $\Phi(t) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{1}{k!}A^k t^k + \cdots$

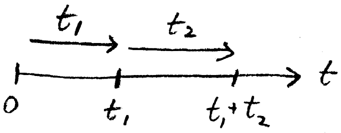
set $t=0 \Rightarrow \Phi(0) = I$

2. Since $\Phi(t) = e^{At}$

$$\Phi(t)e^{-At} = e^{At}e^{-At} = I$$

so $\Phi^{-1}(t)\,\Phi(t)\,e^{-At} = \Phi^{-1}(t)\,I \quad \Rightarrow \quad \Phi^{-1}(t) = e^{-At} = \Phi(-t)$

3.



$\vec{x}(t_1) = e^{At_1}\vec{x}(0)$

and $\vec{x}(t_1+t_2) = e^{At_2}\vec{x}(t_1) = e^{At_2}e^{At_1}\vec{x}(0)$

Also $\vec{x}(t_1+t_2) = e^{A(t_1+t_2)}\vec{x}(0)$

so $e^{A(t_1+t_2)}\vec{x}(0) = e^{At_2}e^{At_1}\vec{x}(0) \rightarrow$ must hold for any $\vec{x}(0)$

$\Rightarrow e^{A(t_1+t_2)} = e^{At_2}e^{At_1} \quad \Rightarrow \quad \Phi(t_1+t_2) = \Phi(t_2)\,\Phi(t_1)$

b) $A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$ $\quad [sI-A] = \begin{bmatrix} s & -1 \\ 5 & s+2 \end{bmatrix}$ $\quad [sI-A]^{-1} = \frac{1}{s(s+2)+5}\begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix}$

$$= \frac{1}{s^2+2s+5}\begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix}$$

$\Phi(t) = \mathcal{L}^{-1}\{[sI-A]^{-1}\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix}\right\} = \begin{bmatrix} e^{-t}(\cos 2t + \frac{1}{2}\sin 2t) & \frac{1}{2}e^{-t}\sin 2t \\ -\frac{5}{2}e^{-t}\sin 2t & e^{-t}(\cos 2t - \frac{1}{2}\sin 2t) \end{bmatrix}$$

$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

$\Phi(-t) = \begin{bmatrix} e^{t}(\cos 2t - \frac{1}{2}\sin 2t) & -\frac{1}{2}e^{t}\sin 2t \\ +\frac{5}{2}e^{t}\sin 2t & e^{t}(\cos 2t + \frac{1}{2}\sin 2t) \end{bmatrix}$ $\quad \Phi(t)\,\Phi(-t) = \begin{bmatrix} \cos^2 2t + \sin^2 2t & 0 \\ 0 & \cos^2 2t + \sin^2 2t \end{bmatrix}$

$= I$ ✓

(a) $\vec{x}(t) = \Phi(t)\vec{x}(0) + \int_0^t \phi(t-\tau) B u(\tau)\, d\tau$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} \phi(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + e^{-2(t-\tau)} \end{bmatrix}_{\tau=0}^{\tau=t}$$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 2e^{-t} - \frac{3}{2} e^{-2t} \\ -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

Since $\vec{y} = \vec{x}$, $\quad \vec{y} = \begin{bmatrix} \frac{1}{2} + 2e^{-t} - \frac{3}{2} e^{-2t} \\ -2e^{-t} + 3e^{-2t} \end{bmatrix}$

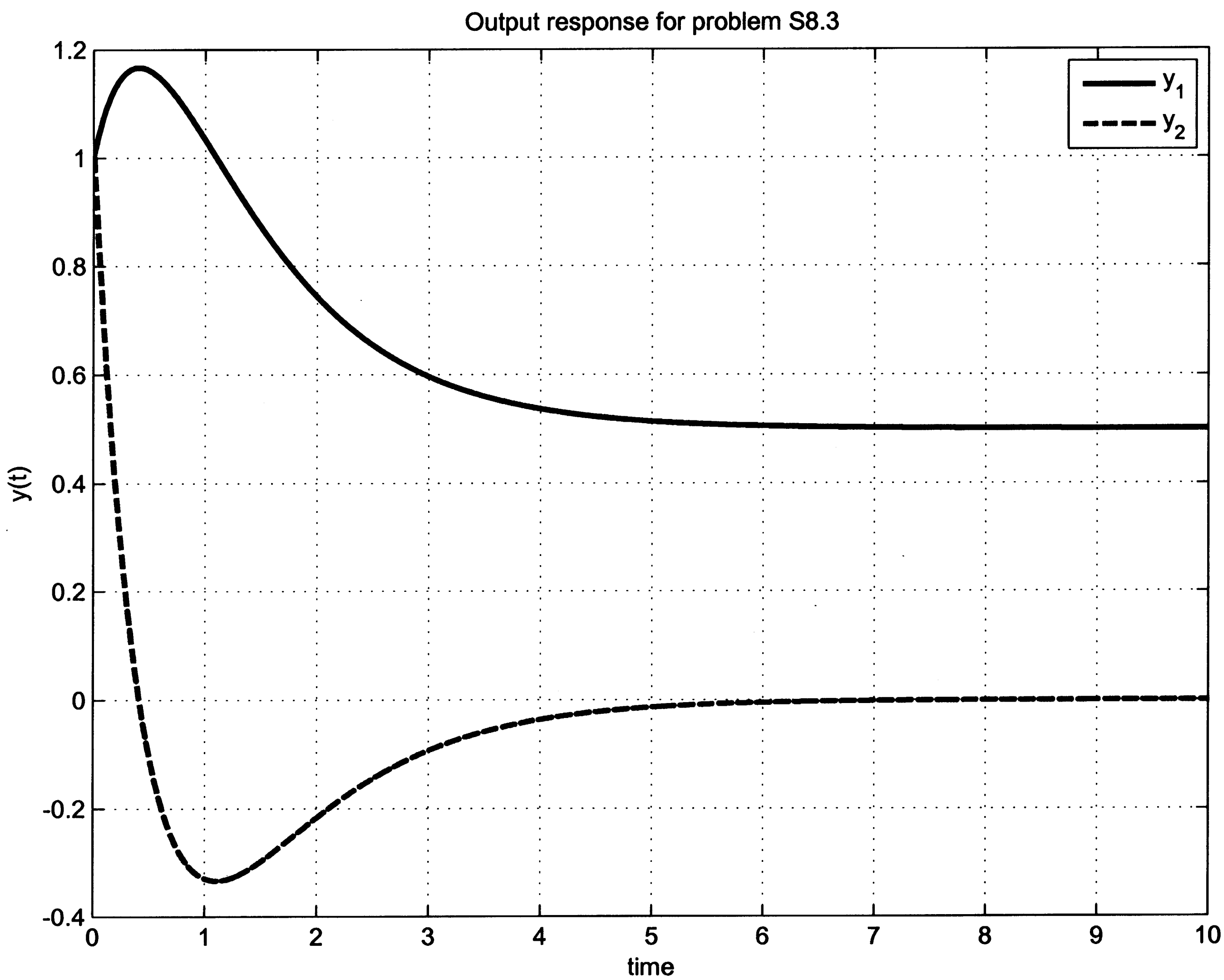
(b) Since $\Phi(t)$ has terms e^{-t}, e^{-2t} $\Rightarrow$ eigenvalues of A are $\lambda = -1$, $\lambda = -2$

Check: $\det[\lambda I - A] = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} = \lambda(\lambda+3) + 2 = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda+1)(\lambda+2) = 0$$

$\Rightarrow$ Overdamped system

(c) See attached plot


Output response for problem S8.3
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
y(t)
0
1
2
3
4
5
6
7
8
9
10
time
y1
y2